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# Convergence Theorems for the Itô-Henstock Integrable Operator-Valued Stochastic Process

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## ABSTRACT

In this paper, we formulate versions of convergence theorems for the Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Wiener process. We also prove that every Itô integrable operator-valued stochastic process is Itô-Henstock integrable using some versions of convergence theorems established in this paper.

**Keywords:** Itô-Henstock integrable, Itô integral, $Q$-Wiener process.

# 1. Introduction and Preliminaries

The Henstock integral, which was studied independently by Henstock and Kurzweil in the 1950s and later known as the Henstock-Kurzweil integral, is one of the notable integrals that was introduced which in some sense is more general than the Lebesgue integral. To avoid an extensive study of measure theory, Henstock-Kurzweil integration had been deeply studied and investigated by numerous authors, see Gordon (1994), Henstock (1988), Kurzweil (2000), Lee (1989), Lee and Výborný (2000), Lee (2011). The Henstock-Kurzweil integral is a Riemann-type definition of an integral which is more explicit and minimizes the technicalities in the classical approach of the Lebesgue integral. This approach to integration is known as the generalized Riemann approach or Henstock approach.

In stochastic calculus, it is not possible to formulate the stochastic integral by means of Riemann approach for the reason that the integrands are highly oscillating and the paths of the integrators are not of bounded variation. For the same reason, it is not even possible to define the stochastic integral as a Riemann-Stieltjes integral, see Mikosch (1998). In the typical approach of stochastic integration, the stochastic integral of a real-valued stochastic process, which is adapted to a filtration, is attained from a limit of stochastic integrals of simple processes. This approach is almost similar in defining the Lebesgue integral of a measurable function. Hence, to give a more explicit definition and reduce the technicalities in the classical way of defining the stochastic integral in the real-valued case, Henstock approach to stochastic integration had already been studied in several papers, see Chew et al. (2003), McShane (1969), Pop-Stojanovic (1972), Toh and Chew (2003, 2005).

In infinite dimensional spaces, the stochastic integral of an operator-valued stochastic process, adapted to a normal filtration, is obtained by extending an isometry from the space of elementary processes to the space of continuous square-integrable martingales. In this case, the value of the integrand is a bounded linear operator and the integrator is a $Q$-Wiener process, a Hilbert space-valued Wiener process which is dependent on a nonnegative, symmetric, and trace-class operator $Q$. This approach requires a deep study of measure theory and functional analysis. In Labendia et al. (2018), the authors introduced a new approach to stochastic integrals in infinite dimensional space and defined the Itô-Henstock integral of an operator-valued stochastic process with respect to a $Q$-Wiener process and formulated a version of Itô's formula, the stochastic counterpart of the classical chain rule of differentiation.

In this paper, we revisit the concept of Itô-Henstock integral for the operator-valued stochastic process with respect to a $Q$-Wiener process and establish versions of convergence theorems. We then use some of the convergence theorems formulated in this paper to show that the classical Itô integral of an operator-valued stochastic process can be defined using Henstock approach.

Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a *filtered probability space*, $\mathcal{B}(H)$ be the Borel $\sigma$-field of a Banach space $H$ which is also separable, and $\mathcal{L}(h)$ be the *probability distribution* or the *law* of a random variable $h : \Omega \to H$.

A *stochastic process* $f : [0,T] \times \Omega \to H$, or simply a *process* $\{f_t\}_{0 \leq t \leq T}$, is said to be *adapted* to a filtration $\{\mathcal{F}_t\}$ if $f_t$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$. When no confusion arises, we may refer to a process adapted to $\{\mathcal{F}_t\}$ as simply an adapted process.

Let $U$ and $V$ be separable Hilbert spaces. Denote by $L(U,V)$ the space of all bounded linear operators from $U$ to $V$, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from $\Omega$ to $V$. The *adjoint* or *dual* $Q^*$ of an operator $Q \in L(U)$ is the unique map $Q^* \in L(U)$ such that $\langle Q^*u, u' \rangle_U = \langle u, Qu' \rangle_U$ for all $u, u' \in U$. An operator $Q \in L(U)$ is said to be *self-adjoint* or *symmetric* if for all $u, u' \in U$, $\langle Qu, u' \rangle_U = \langle u, Qu' \rangle_U$ and is said to be *nonnegative* if for every $u \in U$, $\langle Qu, u \rangle_U \geq 0$. Using the Square-root Lemma (Reed and Simon, 1980, p.196), if $Q \in L(U)$ is nonnegative, then there exists a unique operator $Q^{\frac{1}{2}} \in L(U)$ such that $Q^{\frac{1}{2}}$ is nonnegative and $(Q^{\frac{1}{2}})^2 = Q$.

Let $\{e_j\}_{j=1}^{\infty}$, or simply $\{e_j\}$, be an orthonormal basis (abbrev. as ONB) in $U$. If $Q \in L(U)$ is nonnegative, then the trace of $Q$ is defined by tr $Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_U$. It is shown in (Reed and Simon, 1980, p.206) that tr $Q$ is well-defined and independent of the choice of ONB. An operator $Q : U \to U$ is said to be *trace-class* if tr $[Q] :=$ tr $(QQ^*)^{\frac{1}{2}} < \infty$. Denote by $L_1(U)$ the space of all trace-class operators on $U$, which is known (Reed and Simon, 1980, p.209) to be a Banach space with norm $\|Q\|_1 =$ tr $[Q]$. If $Q \in L(U)$ is a nonnegative, symmetric, and trace-class operator, then there exists an ONB $\{e_j\} \subset U$ and a sequence $\{\lambda_j\}$, $\lambda_j > 0$ $\forall j \in \mathbb{N}$, such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$ and $\lambda_j \to 0$ as $j \to \infty$ (Reed and Simon, 1980, p.203). We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an *eigensequence defined by* $Q$.

Let $Q : U \to U$ be either nonnegative symmetric trace-class operator or $Q = 1_U$, where $1_U$ is the identity function on $U$. If $Q$ is a trace-class operator, let $\{\lambda_j, e_j\}$ be an eigensequence defined by $Q$. Then the subspace $U_Q := Q^{\frac{1}{2}}U$ of $U$ equipped with the inner product $\langle u, v \rangle_{U_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, e_j \rangle_U \langle v, e_j \rangle_U$ is a separable Hilbert space with $\{\sqrt{\lambda_j}e_j\}$ as its ONB, see (Da Prato and Zabczyk, 1992, p.90), (Gawarecki and Mandrekar, 2011, p.23).

Let $\{f_j\}$ be an ONB in $U_Q$. An operator $S \in L(U_Q, V)$ is said to be *Hilbert-Schmidt* if $\sum_{j=1}^{\infty} \|Sf_j\|_V^2 = \sum_{j=1}^{\infty} \langle Sf_j, Sf_j\rangle_V < \infty$. Denote by $L_2(U_Q, V)$ the space of all Hilbert-Schmidt operators from $U_Q$ to $V$, which is known (Prévôt and Röckner, 2007, p.112) to be a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ are independent of the choice of the ONB, see (Da Prato and Zabczyk, 1992, p.418), (Prévôt and Röckner, 2007, p.111). It is shown in (Gawarecki and Mandrekar, 2011, p.25) that $L(U, V)$ is properly contained in $L_2(U_Q, V)$.

Let $Q : U \to U$ be a nonnegative, symmetric, and trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by $Q$, and $\{B_j\}$ be a sequence of independent *Brownian motions* (abbrev. as $BM$). The process

$$\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j \tag{1}$$

is called a *Q-Wiener process in U*. The series in (1) converges in $L^2(\Omega, U)$. For each $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u\rangle_U$, where the series converges in $L^2(\Omega, \mathbb{R})$. By the Strong law of large numbers (Ghahramani, 2005, p.489), it does not necessarily follow that there exists a $U$-valued process $W$ with

$$\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u\rangle_U \quad \mathbb{P}\text{-almost surely (abbrev. as } \mathbb{P}\text{-a.s.).} \tag{2}$$

However, given a nonnegative, symmetric, and trace-class operator $Q$, a $U$-valued process satisfying (2) can be defined. We call the process $W$ a *U-valued Q-Wiener process*. This process is an extension of the $BM$. It should be noted that $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \ldots$, is a sequence of real-valued $BM$, see (Da Prato and Zabczyk, 1992, p.87).

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if (i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and (ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$. A $Q$-Wiener process $W_t$, $t \in [0, T]$ is called a *Q-Wiener process with respect to a filtration* if (i) $W_t$ is adapted to $\{\mathcal{F}_t\}$, $t \in [0, T]$ and (ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t \le T$. It is shown in (Prévôt and Röckner, 2007, p.16) that a $U$-valued $Q$-Wiener process $W(t)$,

$t \in [0,T]$, is a $Q$-Wiener process with respect to a normal filtration. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

An adapted process $M : [0,T] \times \Omega \to V$ is said to be a *martingale* if (i) for all $t \in [0,T]$, $M_t$ is Bochner integrable, i.e. $\mathbb{E}\left[\|M_t\|_V\right] < \infty$ and (ii) for any $0 \le s \le t \le T$, $\mathbb{E}\left[M_t \middle| \mathcal{F}_s\right] = M_s$ $\mathbb{P}$-a.s.. A martingale $M : [0,T] \times \Omega \to V$ is said to be *square-integrable* if $M_T \in L^2(\Omega, V)$. It is known (Prévôt and Röckner, 2007, p.21) that the space of all continuous square-integrable martingales $\mathcal{M}_T^2$ is a Banach space with norm $\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|M_t\|_V^2\right]\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\|M_T\|_V^2\right]\right)^{\frac{1}{2}}$, and the $Q$-Wiener process $W \in \mathcal{M}_T^2$.

An adapted process $f : [0,T] \times \Omega \to L(U,V)$ is called an *elementary process* if there is a finite sequence $\{t_j\}_{j=0}^n$, $n \in \mathbb{N}$, with $0 = t_0 < t_1 < \cdots < t_n = T$ and a finite sequence of random variables $\varphi$, $\{\varphi_j\}$, $j = 0, 1, \dots, n-1$, such that (i) $\varphi : (\Omega, \mathcal{F}_0) \to (L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$ and $\varphi_j : (\Omega, \mathcal{F}_{t_j}) \to (L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$ are measurable with $\varphi(\omega), \varphi_j(\omega) \in L(U,V)$; and (ii) $f(t,\omega) = \varphi(\omega)1_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j(\omega) 1_{(t_j, t_{j+1}]}(t)$. Denote by $\mathcal{E} := \mathcal{E}(U,V)$ the space of all elementary processes. We say that an elementary process $f$ is *bounded* if there exists $M > 0$ such that $\|f(t,\omega)\|_{L_2(U_Q,V)} \le M$ for all $(t,\omega) \in [0,T] \times \Omega$. Denote by $\Lambda_0$ the space of all bounded elementary processes. Then the *Itô integral* of an elementary process $f$ with respect to $W$ is defined by

$$(\mathcal{I}) \int_0^t f_s \, dW_s := \sum_{j=0}^{n-1} \varphi_j (W_{t_{j+1}\wedge t} - W_{t_j \wedge t}) \quad \text{for } t \in [0,T].$$

It is shown in (Prévôt and Röckner, 2007, p.23) that if $f \in \mathcal{E}$, then the Itô integral $(\mathcal{I}) \int_0^t f_s \, dW_s \in \mathcal{M}_T^2$.

The next result is called the *Itô isometry* on $\mathcal{E}$ which gives the norm $\|\cdot\|_{\mathcal{E}}$ on $\mathcal{E}$. It should be noted that the integral on the right hand side of the equality is a Lebesgue integral, indicated by $(\mathcal{L})$.

**Proposition 1.1.** (Prévôt and Röckner, 2007, Proposition 2.3.5). *If $f \in \mathcal{E}$, then*

$$\left\| (\mathcal{I}) \int_0^{(\cdot)} f_s \, dW_s \right\|_{\mathcal{M}_T^2}^2 = \mathbb{E}\left[ (\mathcal{L}) \int_0^T \|f_s\|_{L_2(U_Q,V)}^2 \, ds \right] < \infty.$$

For $f \in \mathcal{E}$, let $\|f\|_{\mathcal{E}} := \sqrt{\mathbb{E}\left[(\mathcal{L})\int_0^T \|f_s\|^2_{L_2(U_Q,V)}\ ds\right]}$. We then construct an equivalence class with respect to $\|\cdot\|_{\mathcal{E}}$ and identify $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ as a normed space, see (Royden and Fitzpatrick, 2007, p.394). In view of Proposition 1.1, for $f \in \mathcal{E}$, $\sqrt{\mathbb{E}\left[(\mathcal{L})\int_0^T \|f_s\|^2_{L_2(U_Q,V)}\ ds\right]} = \|f\|_{\mathcal{E}}$ so that the mapping from $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ to $(\mathcal{M}^2_T, \|\cdot\|_{\mathcal{M}^2_T})$ is an isometry. This mapping can be uniquely extended to a bounded operator from $(\overline{\mathcal{E}}, \|\cdot\|_{\mathcal{E}})$ to $(\mathcal{M}^2_T, \|\cdot\|_{M^2_T})$, which is also an isometry.

Let $\Lambda_2 := \Lambda_2(U_Q, V)$ be the space of all processes $f : [0,T]\times\Omega \to L_2(U_Q, V)$ such that

(i) $f : ([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F}) \to (L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$ is measurable;

(ii) $f$ is adapted to $\{\mathcal{F}_t\}$; and

(iii) $\|f\|_{\mathcal{E}} = \sqrt{\mathbb{E}\left[(\mathcal{L})\int_0^T \|f_s\|^2_{L_2(U_Q,V)}\ ds\right]} < \infty$.

Hence, $\Lambda_0 \subset \mathcal{E} \subset \Lambda_2$. It is shown in (Gawarecki and Mandrekar, 2011, p.28) that $(\Lambda_2, \|\cdot\|_{\mathcal{E}})$ is a Hilbert space.

The next result implies that $\mathcal{E}$ is dense in $\Lambda_2$ so that $\overline{\mathcal{E}} = \Lambda_2$.

**Proposition 1.2.** (Gawarecki and Mandrekar, 2011, Proposition 2.2) *If $f \in \Lambda_2$, then there exists a sequence $\{f^{(n)}\}$, $f^n \in \Lambda_0$, with*

$$\left\|f^{(n)} - f\right\|^2_{\mathcal{E}} = \mathbb{E}\left[(\mathcal{L})\int_0^T \left\|f^{(n)}_t - f_t\right\|^2_{L_2(U_Q,V)}\ dt\right] \to 0 \text{ as } n \to \infty.$$

A stochastic process $f : [0,T]\times\Omega \to L_2(U_Q, V)$ is said to be *Itô integrable* if $f \in \Lambda_2$ and the *Itô integral* of $f$ with respect to $W$ is the unique isometric linear extension of the mapping

$$f(\cdot) \to (\mathcal{I})\int_0^T f_s\ dW_s$$

from the class of bounded elementary processes to $L^2(\Omega, V)$, to a mapping from $\Lambda_2$ to $L^2(\Omega, V)$, such that the image of $f(t,\omega) = \varphi(\omega)\chi_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j(\omega)\chi_{(t_j,t_{j+1}]}(t)$ is $\sum_{j=0}^{n-1} \varphi_j(W_{t_{j+1}} - W_{t_j})$. We define the Itô integral process $(\mathcal{I})\int_0^t$, $0 \leq t \leq T$, for $f \in \Lambda_{\mathcal{I}}$ by

$$(\mathcal{I})\int_0^t f_s \; dW_s = (\mathcal{I})\int_0^T f_s \chi_{[0,t]}(s) \; dW_s.$$

**Theorem 1.1.** (Gawarecki and Mandrekar, 2011, Theorem 2.3) *The stochastic integral $f \to (\mathcal{I})\int_0^{(\cdot)} f_s \; dW_s$ with respect to $W$ is an isometry between $\Lambda_2$ and the space of continuous square-integrable martingales $\mathcal{M}_T^2$,*

$$\mathbb{E}\left[\left\|(\mathcal{I})\int_0^t f_s \; dW_s\right\|_V^2\right] = \mathbb{E}\left[(\mathcal{L})\int_0^t \|f_s\|_{L_2(U_Q,V)}^2 \; ds\right] < \infty$$

*for $t \in [0, T]$.*

## 2. Itô-Henstock Integral and Versions of Convergence Theorems

In this section, assume that $U$ and $V$ are separable Hilberts spaces, $Q : U \to U$ is a nonnegative, symmetric, and trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by $Q$, and $W$ is a $U$-valued $Q$-Wiener process. A stochastic process $f : [0,T] \times \Omega \to L(U, V)$ means a process measurable as mappings from $[0,T] \times \Omega, \mathcal{B}([0,T]) \times \mathcal{F})$ to $(L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$.

Throughout this paper, the given closed interval $[0, T]$ is *nondegenerate*, i.e. $0 < T$ and can be replaced with any closed interval $[a, b]$. If no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection $D$. We shall use the same definition of belated partial division employed by the authors in Chew et al. (2003) to define the Itô-Henstock integral of an $L(U, V)$-valued stochastic process with respect to a $U$-valued $Q$-Wiener process.

**Definition 2.1.** Let $\delta : [0,T] \to (0, \infty)$. A finite collection $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ of interval-point pairs is a *$\delta$-fine belated partial division* of $[0, T]$ if

(i) $\{(\xi_i, v_i]\}_{i=1}^n$ is a disjoint collection of subintervals in $[0, T]$; and

(ii) for all $i \in \{1, 2, \ldots, n\}$, $(\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.

The term *partial* is used in Definition 2.1 since the finite collection of disjoint left-open subintervals of $[0, T]$ may not cover the entire interval $[0, T]$. Using the Vitali covering lemma, the following concept can be defined.

**Definition 2.2.** Given $\eta > 0$, a given $\delta$-fine belated partial division $D = \{((\xi, v], \xi)\}$ is said to be a $(\delta, \eta)$*-fine belated partial division* of $[0, T]$ if

$$\left| T - (D) \sum (v - \xi) \right| \leq \eta.$$

This type of partial division is the basis to which we define the Itô-Henstock integral.

**Definition 2.3.** An adapted process $f : [0, T] \times \Omega \to L(U, V)$ is said to be *Itô-Henstock integrable*, or $\mathcal{IH}$-integrable, on $[0, T]$ with respect to $W$ if there exists $A \in L^2(\Omega, V)$ such that for every $\epsilon > 0$, there is a $\delta : [0, T] \to (0, \infty)$ and a number $\eta > 0$ such that for any $(\delta, \eta)$-fine belated partial division $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^{n}$ of $[0, T]$, we have

$$\mathbb{E}\left[ \| S(f, D, \delta, \eta) - A \|_V^2 \right] < \epsilon,$$

where

$$S(f, D, \delta, \eta) := (D) \sum f_\xi (W_v - W_\xi) := \sum_{i=1}^{n} f_{\xi_i} (W_{v_i} - W_{\xi_i}).$$

In this case, $f$ is $\mathcal{IH}$-integrable to $A$ on $[0, T]$ and $A$ is called the $\mathcal{IH}$-integral of $f$ which will be denoted by $(\mathcal{IH}) \displaystyle\int_0^T f_t \, dW_t$ or $(\mathcal{IH}) \displaystyle\int_0^T f \, dW$.

We note that the given closed and bounded interval $[0, T]$ is *nondegenerate*, i.e. $0 < T$. For convenience, we shall denote $(\mathcal{IH}) \displaystyle\int_0^0 f_t \, dW_t$ by the zero random variable $\mathbf{0} \in L^2(\Omega, V)$.

**Example 2.1.** *Let $f : [0, T] \times \Omega \to L(U, V)$ be an adapted process with $\mathbb{E}\left[ \|f_t\|_{L_2(U_Q, V)}^2 \right] = 0$ almost everywhere (abbrev. as a.e.) on $[0, T]$. Then $f$ is $\mathcal{IH}$-integrable to $\mathbf{0}$ on $[0, T]$.*

It is worth noting that the Itô-Henstock integral possesses the standard properties of an integral namely, uniqueness, linearity, integrability on every subinterval of $[0,T]$, and Cauchy criterion. The proofs of the following results are standard in Henstock-Kurzweil integration, hence omitted.

**Theorem 2.1.** *A process $f:[0,T]\times\Omega\to L(U,V)$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if there exist $A\in L^2(\Omega,V)$, a decreasing sequence $\{\delta_n(\xi)\}$, $\delta_n:[0,T]\to(0,\infty)$, and a decreasing sequence $\{\eta_n\}$, $\eta_n>0$, such that for any $(\delta_n,\eta_n)$-fine belated partial division $D_n$ of $[0,T]$, we have*

$$\mathbb{E}\left[\|S(f,D_n,\delta_n,\eta_n)-A\|_V^2\right]\to 0 \text{ as } n\to\infty.$$

*In this case, $A=(\mathcal{IH})\int_0^T f_t\, dW_t$.*

**Lemma 2.1.** (Henstock Lemma (Weak Version)). *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and $F(u,v):=(\mathcal{IH})\int_u^v f_t\, dW_t$ for any $(u,v]\subset[0,T]$. Then for every $\epsilon>0$, there exists a $\delta:[0,T]\to(0,\infty)$ such that whenever $D=\{((\xi,v],\xi)\}$ is a $\delta$-fine belated partial division of $[0,T]$, we have*

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v-W_\xi)-F(\xi,v)\}\right\|_V^2\right]<\epsilon.$$

**Theorem 2.2.** (Itô Isometry). *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and*

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\, dW_t\right\|_V^2\right]=(\mathcal{L})\int_0^T\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\,dt<\infty.$$

Next, we characterize the Itô-Henstock integral using $AC^2[0,T]$-property, a version of absolute continuity.

**Definition 2.4.** A process $F:[0,T]\times\Omega\to V$ is said to be $AC^2[0,T]$ if for every $\epsilon>0$, there exists $\eta>0$ such that for any finite collection $D=\{(\xi,v]\}$ of disjoint subintervals of $[0,T]$ with $(D)\sum(v-\xi)<\eta$, we have

$$\mathbb{E}\left[\left\|(D)\sum(F_v-F_\xi)\right\|_V^2\right]:=\int_\Omega\left\|(D)\sum(F(v,\omega)-F(\xi,\omega))\right\|_V^2\,d\mathbb{P}<\epsilon.$$

It is not difficult to show that the Itô-Henstock integral is $AC^2[0,T]$.

**Theorem 2.3.** *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and define*

$$F_t := (\mathcal{IH}) \int_0^t f_s \, dW_s \text{ for all } t \in [0,T].$$

*Then $F$ is $AC^2[0,T]$.*

**Theorem 2.4.** *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process. Then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if there exists a process $F : [0,T] \times \Omega \to V$ such that*

*(i) $F$ is $AC^2[0,T]$ and*

*(ii) for every $\epsilon > 0$, there exists a $\delta : [0,T] \to (0,\infty)$ such that whenever $D = \{((\xi,v],\xi)\}$ is a $\delta$-fine belated partial division of $[0,T]$, we have*

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v - W_\xi) - (F_v - F_\xi)\}\right\|_V^2\right] < \epsilon.$$

***Proof.*** Suppose that $f$ is $\mathcal{IH}$-integrable. Then (i) and (ii) hold by Theorem 2.3 and the weak version of Henstock lemma.

Conversely, assume that (i) and (ii) hold. Let $\epsilon > 0$ be given. Since $F$ is $AC^2[0,T]$, choose $\eta > 0$ such that whenever $\{(\xi_j, v_j]\}_{j=1}^m$ is a finite collection of subintervals $(\xi_j, v_j] \subset [0,T]$ with $\sum_{j=1}^{m} |v_j - \xi_j| < \eta$ we have

$$\mathbb{E}\left[\left\|\sum_{j=1}^{m}(F_{v_j} - F_{\xi_j})\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Let $D = \{((\xi,v],\xi)\}$ be a $(\delta,\eta)$-fine belated partial division of $[0,T]$ and let $D^c$ be the collection of all subintervals of $[0,T]$ which are not included in the set $D$. Since $F$ is $AC^2[0,T]$,

$$\mathbb{E}\left[\left\|(D^c)\sum(F_v - F_\xi)\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_\xi) - (F_T - F_0)\right\|_V^2\right] & \\
\leq\ & 2\mathbb{E}\left[\left\|(D)\sum \{f_\xi(W_v - W_\xi) - (F_v - F_\xi)\}\right\|_V^2\right] \\
& +2\mathbb{E}\left[\left\|(D^c)\sum (F_v - F_\xi)\right\|_V^2\right] \\
<\ & \epsilon.
\end{aligned}$$

Thus, $f$ is $\mathcal{IH}$-integrable on $[0,T]$. □

**Lemma 2.2.** *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and define*

$$F(\xi, v) := (\mathcal{IH})\int_\xi^v f_t \; dW_t$$

*for all $(\xi, v] \subset [0,T]$. Then for any disjoint intervals $(a,b], (u,v] \subset [0,T]$,*

*(i) $F$ has the orthogonal increment property, i.e.* $\mathbb{E}\left[\langle F(a,b), F(u,v)\rangle_V\right] = 0$;

*(ii)* $\mathbb{E}\left[\langle f_a(W_b - W_a), F(u,v)\rangle_V\right] = 0$.

***Proof.*** (i) Assume that $b \leq u$. By Theorem 2.1, there exist a decreasing sequence $\{\delta_n(\xi)\}$, $\delta_n : [0,T] \to (0,\infty)$ and a decreasing sequence $\{\eta_n\}$, $\eta_n > 0$, such that for any $(\delta_n, \eta_n)$-fine belated partial divisions $D_n(a,b) = \{((\xi_i, v_i], \xi_i)\}_{i=1}^m$ and $D_n(u,v) = \{((\xi_j, v_j], \xi_j)\}_{j=1}^p$ of $[a,b]$ and $[u,v]$, respectively, we have

$$\mathbb{E}\left[\|S(f, D_n(a,b), \delta_n, \eta_n) - F(a,b)\|_V^2\right] \to 0 \text{ as } n \to \infty$$

and

$$\mathbb{E}\left[\|S(f, D_n(u,v), \delta_n, \eta_n) - F(u,v)\|_V^2\right] \to 0 \text{ as } n \to \infty.$$

From (Labendia et al., 2018, Lemma 3.5), for every $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\langle S(f, D_n(a,b), \delta_n, \eta_n), S(f, D_n(u,v), \delta_n, \eta_n)\rangle_V\right] \\
&\qquad = \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^p \left\langle f_{\xi_i}(W_{v_i} - W_{\xi_i}), f_{\xi_j}(W_{v_j} - W_{\xi_j})\right\rangle_V\right] = 0.
\end{aligned}$$

Since $S(f, D_n(a,b), \delta_n, \eta_n) \to F(a,b)$ and $S(f, D_n(u,v), \delta_n, \eta_n) \to F(u,v)$ in $L^2(\Omega, V)$ as $n \to \infty$, it follows that

$$\mathbb{E}\left[\langle S(f, D_n(a,b), \delta_n, \eta_n), S(f, D_n(u,v), \delta_n, \eta_n)\rangle_V\right] \to \mathbb{E}\left[\langle F(a,b), F(u,v)\rangle_V\right]$$

as $n \to \infty$, see (Zeidler, 1990, p.413). Thus, $\mathbb{E}\left[\langle F(a,b)F(u,v)\rangle_V\right] = 0$.

(ii) Note that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\left[\langle f_a(W_b - W_a), S(f, D_n(u,v), \delta_n, \eta_n)\rangle_V\right] \\
&\quad = \quad \mathbb{E}\left[\sum_{j=1}^{p} \left\langle f_a(W_b - W_a), f_{\xi_j}(W_{v_j} - W_{\xi_j})\right\rangle_V\right] = 0.
\end{aligned}$$

Since $f_a(W_b - W_a) \to f_a(W_b - W_a)$ and $S(f, D_n(u,v), \delta_n, \eta_n) \to F(u,v)$ in $L^2(\Omega, V)$ as $n \to \infty$, it follows that

$$\mathbb{E}\left[\langle f_a(W_b - W_a), S(f, D_n(u,v), \delta_n, \eta_n)\rangle_V\right] \to \mathbb{E}\left[\langle f_a(W_b - W_a), F(u,v)\rangle_V\right]$$

as $n \to \infty$. Thus $\mathbb{E}\left[\langle f_a(W_b - W_a), F(u,v)\rangle_V\right] = 0$. □

**Lemma 2.3.** *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and define*

$$F(\xi, v) := (\mathcal{IH}) \int_{\xi}^{v} f_t \, dW_t$$

*for all $(\xi, v] \subset [0,T]$. Let $\{(\xi_i, v_i]\}_{i=1}^{n}$ be a finite disjoint collection of subintervals in $[0,T]$. Then*

*(i)* $$\mathbb{E}\left[\left\|\sum_{i=1}^{n} f_\xi(W_v - W_\xi)\right\|_V^2\right] = \sum_{i=1}^{n}(v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right];$$

*(ii)* $$\mathbb{E}\left[\left\|\sum_{i=1}^{n}\{f_\xi(W_v - W_\xi) - F(\xi,v)\}\right\|_V^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi,v)\|_V^2\right];$$

*(iii)* $$\mathbb{E}\left[\left\|\sum_{i=1}^{n} F(\xi,v)\right\|_V^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[\|F(\xi,v)\|_V^2\right].$$

***Proof.*** (i) This is immediate from (Labendia et al., 2018, Lemma 3.6).

(ii) Let $\{(\xi_i, v_i]\}_{i=1}^{n}$ be a finite disjoint collection of subintervals in $[0,T]$. Then by Lemma 2.2, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{i=1}^{n}\{f_{\xi_i}(W_{v_i}-W_{\xi_i})-F(\xi_i,v_i)\}\right\|_V^2\right]\\
= &\sum_{i=1}^{n}\mathbb{E}\left[\|f_{\xi_i}(W_{v_i}-W_{\xi_i})-F(\xi_i,v_i)\|_V^2\right]\\
&\quad +2\sum_{i<j}\mathbb{E}\left[\left\langle f_{\xi_i}(W_{v_i}-W_{\xi_i})-F(\xi_i,v_i), f_{\xi_j}(W_{v_j}-W_{\xi_j})-F(\xi_j,v_j)\right\rangle_V\right]\\
= &\sum_{i=1}^{n}\mathbb{E}\left[\|f_{\xi_i}(W_{v_i}-W_{\xi_i})-F(\xi_i,v_i)\|_V^2\right]\\
&\quad +2\sum_{i<j}\mathbb{E}\left[\left\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}), f_{\xi_j}(W_{v_j}-W_{\xi_j})\right\rangle_V\right]\\
&\qquad -2\sum_{i<j}\mathbb{E}\left[\langle f_{\xi_i}(W_{v_i}-W_{\xi_i}), F(\xi_j,v_j)\rangle_V\right]\\
&\qquad\quad -2\sum_{i<j}\mathbb{E}\left[\left\langle f_{\xi_j}(W_{v_j}-W_{\xi_j}), F(\xi_i,v_i)\right\rangle_V\right]\\
&\qquad\qquad +2\sum_{i<j}\mathbb{E}\left[\langle F(\xi_i,v_i), F(\xi_j,v_j)\rangle_V\right]\\
= &\sum_{i=1}^{n}\mathbb{E}\left[\|f_{\xi_i}(W_{v_i}-W_{\xi_i})-F(\xi_i,v_i)\|_V^2\right].
\end{aligned}
$$

(iii) Since $F$ has the orthogonal increment property,

$$
\begin{aligned}
\mathbb{E}\left[\left\|\sum_{i=1}^{n}F(\xi_i,v_i)\right\|_V^2\right] &= \sum_{i=1}^{n}\mathbb{E}\left[\|F(\xi_i,v_i)\|_V^2\right]\\
&\quad +2\sum_{i<j}\mathbb{E}\left[\langle F(\xi_i,v_i), F(\xi_j,v_j)\rangle_V\right]\\
&= \sum_{i=1}^{n}\mathbb{E}\left[\|F(\xi_i,v_i)\|_V^2\right],
\end{aligned}
$$

thereby completing the proof. □

The strong version of Henstock lemma follows from Lemma 2.3(ii).

**Lemma 2.4.** (Henstock Lemma (Strong Version)). *Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and $F(u,v) := (\mathcal{IH})\int_u^v f_t\, dW_t$ for any $(u,v] \subset [0,T]$. Then for every $\epsilon > 0$, there exists a $\delta : [0,T] \to (0,\infty)$ such that whenever $D = \{((\xi,v],\xi)\}$ is* *a $\delta$-fine belated partial division of $[0,T]$, we have*

$$(D)\sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi,v)\|_V^2\right] < \epsilon.$$

**Lemma 2.5.** *Let $\{f^{(n)}\}$ be a sequence of $\mathcal{IH}$-integrable processes on $[0,T]$ such that $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t\right\}$ is Cauchy in $L^2(\Omega,V)$. Then for all $t\in[0,T]$, there exists $A_t \in L^2(\Omega,V)$ and the following property is satisfied: for every $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all finite collection $\{(\xi_i,v_i]\}_{i=1}^p$ of disjoint intervals of $[0,T]$,*

$$\mathbb{E}\left[\left\|\sum_{i=1}^p\left\{(\mathcal{IH})\int_{\xi_i}^{v_i} f_t^{(n)}\, dW_t - (A_{v_i}-A_{\xi_i})\right\}\right\|_V^2\right] < \epsilon$$

*whenever $n \geq N$.*

***Proof.*** Let $\epsilon>0$ be given. Since $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t\right\}$ is Cauchy in $L^2(\Omega,V)$, for all $t\in[0,T]$, $\left\{(\mathcal{IH})\int_0^t f_s^{(n)}\, dW_s\right\}$ is also Cauchy in $L^2(\Omega,V)$ by Lemma 2.3(iii). Then for all $t\in[0,T]$, there exists $A_t\in L^2(\Omega,V)$ such that $(\mathcal{IH})\int_0^t f_s^{(n)}\, dW_s \to A_t$ in $L^2(\Omega,V)$ as $n\to\infty$. Let $\{(\xi_i,v_i]\}_{i=1}^p$ be a finite collection of disjoint intervals of $[0,T]$. Then for all $i\in\{1,2,\ldots,p\}$, there exists $N_i\in\mathbb{N}$ such that for all $n\geq N_i$,

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_{\xi_i}^{v_i} f_t^{(n)} - (A_{v_i}-A_{\xi_i})\right\|_V^2\right] < \frac{\epsilon}{2^{2i}}.$$

Take $N=\max\{N_i : 1\leq i\leq p\}$. Hence, for $n\geq N$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{i=1}^p\left\{(\mathcal{IH})\int_{\xi_i}^{v_i} f_t^{(n)}\, dW_t - (A_{v_i}-A_{\xi_i})\right\}\right\|_V^2\right]\\
&\quad\leq \left(\sum_{i=1}^p\sqrt{\mathbb{E}\left[\left\|(\mathcal{IH})\int_{\xi_i}^{v_i} f_t^{(n)}\, dW_t - (A_{v_i}-A_{\xi_i})\right\|_V^2\right]}\right)^2\\
&\quad< \left(\sum_{i=1}^p\frac{\sqrt{\epsilon}}{2^j}\right)^2 \leq \epsilon.
\end{aligned}$$

This completes the proof. □

We now formulate versions of convergence theorems for the $\mathcal{IH}$-integral.

**Theorem 2.5.** (Mean Convergence Theorem). *Let $\{f^{(n)}\}$ be a sequence of $\mathcal{IH}$-integrable processes on $[0,T]$ and $f$ be an adapted process on $[0,T]$ such that*

*(i)* $\mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right] \to 0$ *a.e. on* $[0,T]$;

*(ii)* $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t\right\}$ *is Cauchy in* $L^2(\Omega, V)$.

*Then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and*

$$(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t \to (\mathcal{IH})\int_0^T f_t\, dW_t \text{ as } n\to\infty.$$

***Proof.*** Let $\epsilon > 0$ be given. By Example 2.1, for every $t\in[0,T]$, there exists $N_1(t)\in\mathbb{N}$ such that for all $n\geq N_1(t)$, $\mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|^2_{L_2(U_Q,V)}\right] < \frac{\epsilon}{9T}$. By the weak version of Henstock lemma, for all $n\in\mathbb{N}$, there exists a $\delta:[0,T]\to(0,\infty)$ such that whenever $D=\{((\xi,v],\xi)\}$ is a $\delta$-fine belated partial division of $[0,T]$,

$$\mathbb{E}\left[\left\|(D_n)\sum\left\{f_\xi^{(n)}(W_v - W_\xi) - (\mathcal{IH})\int_\xi^v f_t^{(n)}\, dW_t\right\}\right\|_V^2\right] < \frac{\epsilon}{9}.$$

Since $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t\right\}$ is Cauchy in $L^2(\Omega,V)$, by Lemma 2.5, for all $t\in[0,T]$, there exists $F_t\in L^2(\Omega,V)$ with the following property: there exists $N_2\in\mathbb{N}$ such that for all finite collection $\{(\xi_i,v_i]\}_{i=1}^p$ of disjoint intervals of $[0,T]$,

$$\mathbb{E}\left[\left\|\sum_{i=1}^p\left\{(\mathcal{IH})\int_{\xi_i}^{v_i} f_t^{(n)}\, dW_t - (F_{v_i} - F_{\xi_i})\right\}\right\|_V^2\right] < \frac{\epsilon}{9} \tag{3}$$

whenever $n\geq N_2$. Let $F^{(n)}(\xi,v) := (\mathcal{IH})\int_\xi^v f_t^{(n)}\, dW_t$ for all $(\xi,v]\subset[0,T]$. Choose $N := N(\xi) > \max\{N_1(\xi), N_2\}$ and pick $\delta(\xi)=\delta_N(\xi)$ for all $\xi\in[0,T]$.

Let $D = \{((\xi, v], \xi)\}$ be a $\delta$-fine belated partial division of $[0, T]$. By (Labendia et al., 2018, Lemma 3.6),

$$\begin{aligned} &\mathbb{E}\left[\left\|(D)\sum(f_\xi - f_\xi^{(N)})(W_v - W_\xi)\right\|_V^2\right] \\ &\qquad = (D)\sum(v-\xi)\mathbb{E}\left[\left\|f_\xi - f_\xi^{(N)}\right\|^2_{L_2(U_Q,V)}\right] \\ &\qquad < \frac{\epsilon}{9T}\cdot T = \frac{\epsilon}{9}. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v - W_\xi) - (F_v - F_\xi)\}\right\|_V^2\right] \\ &\quad \leq 3\mathbb{E}\left[\left\|(D)\sum(f_\xi - f_\xi^{(N)})(W_v - W_\xi)\right\|_V^2\right] \\ &\qquad +3\mathbb{E}\left[\left\|(D)\sum\{F^{(N)}(\xi, v) - (F_v - F_\xi)\}\right\|_V^2\right] \\ &\qquad\quad +3\mathbb{E}\left[\left\|(D)\sum\{f_\xi^{(N)}(W_v - W_\xi) - F^{(N)}(\xi, v)\}\right\|_V^2\right] \\ &\quad < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Next, we show that $F$ is $AC^2[0, T]$. By Theorem 2.3, $F^{(N)}$ is $AC^2[0, T]$. Let $\eta > 0$ which responds to $\frac{7\epsilon}{18}$ challenge regarding the criterion for the $AC^2[0, T]$ property of $F^{(N)}$. Let $D' = \{(\xi', v']\}$ be a disjoint collection of subintervals in $[0, T]$ with $(D')\sum(v' - \xi') < \eta$. Then

$$\begin{aligned} &\mathbb{E}\left[\left\|(D')\sum(F_{v'} - F_{\xi'})\right\|_V^2\right] \\ &\qquad \leq 2\mathbb{E}\left[\left\|(D')\sum F^{(N)}(\xi', v')\right\|_V^2\right] \\ &\qquad\quad +2\mathbb{E}\left[\left\|(D')\sum\left\{F^{(N)}(\xi', v') - (F_{v'} - F_{\xi'})\right\}\right\|_V^2\right] \\ &\qquad < 2\left(\frac{7\epsilon}{18}\right) + 2\left(\frac{\epsilon}{9}\right) = \epsilon. \end{aligned}$$

Thus, $F$ is $AC^2[0, T]$. By Theorem 2.4, $f$ is $\mathcal{IH}$-integrable on $[0, T]$.

To show that $(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t \to (\mathcal{IH})\int_0^T f_t\ dW_t$, let $D^c = \{(\xi, v]\}$ be the collection of subintervals of $[0,T]$ which are not included in $D$. From equation (3),

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t - (F_T - F_0)\right\|_V^2\right] < \frac{\epsilon}{9}$$

whenever $n \geq N_2$. From the proof of Theorem 2.4, $(\mathcal{IH})\int_0^T f_t\ dW_t = F_T - F_0$. Thus, $(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t \to (\mathcal{IH})\int_0^T f_t\ dW_t$ as $n \to \infty$. □

**Theorem 2.6.** (Dominated Convergence Theorem). *Let $\{f^{(n)}\}$ be a sequence of $\mathcal{IH}$-integrable processes on $[0,T]$ and $f$ be an adapted process on $[0,T]$ such that*

*(i)* $\mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|_{L_2(U_Q,V)}^2\right] \to 0$ *a.e. on* $[0,T]$;

*(ii) for almost all $t \in [0,T]$ and for almost all $\omega \in \Omega$,*

$$\left\|f_t^{(n)}(\omega)\right\|_{L_2(U_Q,V)} \leq \|g_t(\omega)\|_{L_2(U_Q,V)} \text{ for all } n \in \mathbb{N}$$

*and that $g$ is $\mathcal{IH}$-integrable on $[0,T]$.*

*Then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and*

$$(\mathcal{IH})\int_0^T f_t^{(n)}\ dW_t \to (\mathcal{IH})\int_0^T f_t\ dW_t \text{ as } n \to \infty.$$

***Proof.*** Since $\mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|_{L_2(U_Q,V)}^2\right] \to 0$ a.e. on $[0,T]$,

$$\mathbb{E}\left[\left\|f_t^{(n)}\right\|_{L_2(U_Q,V)}^2\right] \to \mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)}^2\right] \text{ a.e. on } [0,T].$$

By Itô isometry, for each $n \in \mathbb{N}$, $\mathbb{E}\left[\left\|f_t^{(n)}\right\|_{L_2(U_Q,V)}^2\right]$ is Lebesgue integrable on $[0,T]$. Moreover, $\mathbb{E}\left[\|g_t\|_{L_2(U_Q,V)}^2\right]$ is also Lebesgue integrable on $[0,T]$ with

$$\mathbb{E}\left[\left\|f_t^{(n)}\right\|_{L_2(U_Q,V)}^2\right] \leq \mathbb{E}\left[\|g_t\|_{L_2(U_Q,V)}^2\right] \text{ for all } n \in \mathbb{N}.$$

Using the dominated convergence theorem for Lebesgue integral, $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and

$$(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)}\right\|^2_{L_2(U_Q,V)}\right] \to (\mathcal{L})\int_0^T \mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right] \text{ as } n\to\infty.$$

By Itô isometry, $\left\{(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t\right\}$ converges in $L^2(\Omega,V)$. By Theorem 2.5, $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and

$$(\mathcal{IH})\int_0^T f_t^{(n)}\, dW_t \to (\mathcal{IH})\int_0^T f_t\, dW_t \text{ as } n\to\infty.$$

This completes the proof. □

**Theorem 2.7.** *If $f\in\Lambda_0$, then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and*

$$(\mathcal{IH})\int_0^T f_t\, dW_t = (\mathcal{I})\int_0^T f_t\, dW_t.$$

***Proof.*** Let $\epsilon>0$ be given. Let $J$ be a family of all left-open subintervals $(\xi,v]$ of [0,T]. Suppose that $f\in\Lambda_0$ is given by

$$f(t,\omega) = \phi(\omega)1_{\{0\}}(t) + \sum_{j=0}^{n-1}\phi_j(\omega)1_{(t_j,t_{j+1}]}(t).$$

The function $F$ defined by $F(\xi,v) = (\mathcal{I})\int_\xi^v f_t\, dW_t$ for all $(\xi,v]\in J$, is $AC^2[0,T]$. Then there exists $\eta>0$ such that for any finite collection $D=\{(\xi,v]\}$ of disjoint subintervals $(\xi,v]\in J$ with $(D)\sum(v-\xi)<\eta$, we have $\mathbb{E}\left[\left\|(D)\sum F(\xi,v)\right\|_V^2\right] < \frac{\epsilon}{2}$. Let $\xi\in[0,T]$. We shall only consider $\xi\neq t_j$ for all $j=0,1,\ldots,n-1$. Assume that $\xi\in(t_j,t_{j+1})$. Choose $\delta(\xi)>0$ such that $[\xi,\xi+\delta(\xi))\subset(t_j,t_{j+1})$. It follows that $f(\xi,\omega)=\phi_j(\omega)$ and

$$f(\xi,\omega)(W_v(\omega)-W_\xi(\omega)) = \phi_j(\omega)(W_v(\omega)-W_\xi(\omega))$$

for any $(\xi,v]\subset[\xi,\xi+\delta(\xi))$. Moreover, $(\mathcal{I})\int_\xi^v f_t\, dW_t = \phi_j(W_v-W_\xi)$. Let $D=\{((\xi,v],\xi)\}$ be a $(\delta,\eta)$-fine belated partial division of $[0,T]$ with $\xi\neq t_j$ for all $j=0,1,\ldots,n-1$ and $D^c$ be the collection of all subintervals of $[0,T]$ which are not included in the set $D$. Then

$$\mathbb{E}\left[\left\|(D)\sum\left\{f_\xi(W_v-W_\xi)-(\mathcal{I})\int_\xi^v f_t\, dW_t\right\}\right\|_V^2\right] = 0.$$

Moreover,

$$\mathbb{E}\left[\left\|(D^c)\sum\left\{(\mathcal{I})\int_{\xi}^{v} f_t\ dW_t\right\}\right\|_V^2\right] < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_\xi) - (\mathcal{I})\int_0^T f_t\ dW_t\right\|_V^2\right] \\
&\quad\leq 2\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_\xi) - (\mathcal{I})\int_\xi^v f_t\ dW_t\right\|_V^2\right] \\
&\qquad +2\mathbb{E}\left[\left\|(D^c)\sum\left\{(\mathcal{I})\int_\xi^v f_t\ dW_t\right\}\right\|_V^2\right] \\
&\quad< \epsilon.
\end{aligned}$$

Thus, $f$ is $\mathcal{IH}$-integrable to $(\mathcal{I})\displaystyle\int_0^T f_t\ dW_t$ □

The next result shows that if a process $f : [0,T]\times\Omega \to L(U,V)$ is Itô integrable, then it is Itô-Henstock integrable.

**Theorem 2.8.** *If $f \in \Lambda_2$, then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and*

$$(\mathcal{IH})\int_0^T f_t\ dW_t = (\mathcal{I})\int_0^T f_t\ dW_t.$$

***Proof.*** Let $\epsilon > 0$ be given. By Proposition 1.2 and Fubini-Tonelli theorem (Lavrent'ev and Savel'ev, 2006, p.306), there exists a sequence $\{f^{(n)}\}$, $f^n \in \Lambda_0$, such that

$$(\mathcal{L})\int_0^T \mathbb{E}\left[\left\|f_t^{(n)} - f_t\right\|_{L_2(U_Q,V)}^2\right]\ dt \to 0 \text{ as } n\to\infty.$$

By the partial converse of Lebesgue dominated convergence theorem (Uribe and Fiorenza, 2013, p.49), there exists a subsequence $\{f^{(n_m)}\}$ of $\{f^{(n)}\}$ such that

$$\mathbb{E}\left[\left\|f_t^{(n_m)} - f_t\right\|_{L_2(U_Q,V)}^2\right] \to 0 \text{ a.e. on } [0,T].$$

By Proposition 1.2,

$$\mathbb{E}\left[\left\|(\mathcal{I})\int_0^T (f_t^{(n_m)} - f_t)\ dW_t\right\|_V^2\right] = \mathbb{E}\left[(\mathcal{L})\int_0^T \left\|f_t^{(n_m)} - f_t\right\|_{L_2(U_Q,V)}^2\ dt\right] \to 0.$$

By Theorem 2.7, $\left\{(\mathcal{IH}) \int_0^T f_t^{(n_m)} \, dW_t\right\}_{m\in\mathbb{N}}$ converges in $L^2(\Omega, V)$. By Theorem 2.5, $f$ is $\mathcal{IH}$-integrable on $[0, T]$ and in $L^2(\Omega, V)$

$$\begin{aligned}(\mathcal{IH}) \int_0^T f_t \, dW_t &= \lim_{m\to\infty} (\mathcal{IH}) \int_0^T f_t^{(n_m)} \, dW_t \\ &= \lim_{m\to\infty} (\mathcal{I}) \int_0^T f_t^{(n_m)} \, dW_t \\ &= (\mathcal{I}) \int_0^T f_t \, dW_t.\end{aligned}$$

This completes the proof. □

# 3. Conclusion and Recommendation

In this paper, we formulate versions of convergence theorems for the Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued $Q$-Wiener process. We then use some of these theorems to verify that the classical Itô integral of an operator-valued stochastic process can be defined using Henstock approach. A worthwhile direction for further investigation is to use Henstock approach to define the stochastic integral with respect to a cylindrical Wiener process introduced by Riedle (2011).

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